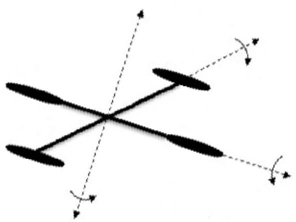
Laplace Transform -2

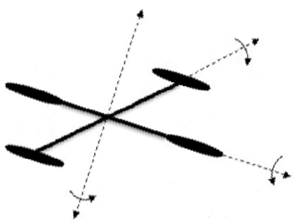


Inverse Laplace

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\,ds$$

A complex integral with σ a value that causes the line $\sigma - j\infty$ to $\sigma + j\infty$ to lie in the ROC.

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathscr{L}^{-1}\{F(s)\} + \beta\mathscr{L}^{-1}\{G(s)\},$$



정리 4.3 기본적인 역변환

(a) $1 = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

(b) $t^n = \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\},\ n = 1, 2, 3, \ldots$ (c) $e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$

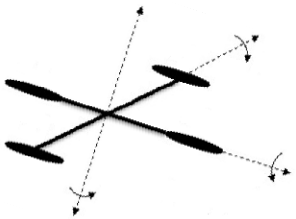
(d) $\sin kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$ (e) $\cos kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathcal{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$ (g) $\cosh kt = \mathcal{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

■ **$\mathcal{L}^{-1}$은 선형변환이다** 역 Laplace 변환도 선형변환이다. 즉 F와 G가 어떤 함수 f와 g의 변환일 때, 상수 α와 β에 대하여

$$\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\} \qquad (1)$$

이 성립한다. 4.1 절의 (2)와 마찬가지로 (1)은 Laplace 변환의 유한 개의 일차결합으로까지 확장될 수 있다.



예제 1 정리 4.3의 적용

다음을 구하라.

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$ (b) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+7}\right\}$

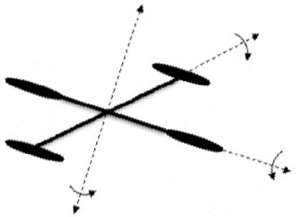
풀이 (a) 정리 4.3의 (b)를 이용하기 위해 $n+1=5$를 택하고 $4!$을 곱하고 나눔으로써

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}=\frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\}=\frac{1}{24}t^4$$

을 얻는다.

(b) 정리 4.3의 (d)를 이용하기 위해 $k^2=7$, 즉 $k=\sqrt{7}$로 한다. $\sqrt{7}$을 곱하고 나눔으로써 아래의 식을 구한다.

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\}=\frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\}=\frac{1}{\sqrt{7}}\sin\sqrt{7}t$$ ❑



예제 2 항별나누기와 선형성

$\mathcal{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$ 을 구하라.

풀이 우선 주어진 s 의 함수를 항별나누기하여 두 부분으로 다시 쓰고 (1)을 이용하자. 그러면 다음과 같이 정리된다.

$$\mathcal{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} \overset{\text{항별나누기}}{=} \mathcal{L}^{-1}\left\{\frac{-2s}{s^2+4}+\frac{6}{s^2+4}\right\} \overset{\text{선형성과 상수조정}}{=} -2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}+\frac{6}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \qquad (2)$$

$$= -2\cos 2t + 3\sin 2t \quad \leftarrow k=2\text{일 때의 정리 4.3의 (e), (d)}$$

❑

예제 3 부분분수와 선형성

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$$ 을 구하라.

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} \pm \frac{C}{s+4}$$

$$= \frac{A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

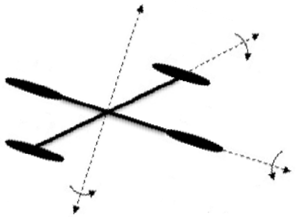
$$s^2+6s+9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

$s=1,\ s=2,\ s=-4$ → $16 = A(-1)(5),\quad 25 = B(1)(6),\quad 1 = C(-5)(-6)$

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$$



Example)

$$P(s) = \frac{9s^2 + 27s + 9.8}{s(s^2 + 5s + 6)} = \frac{C_1}{s} + \frac{C_2}{s+2} + \frac{C_3}{s+3}$$

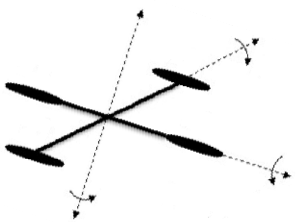
$$C_1 = \left. \frac{9s^2 + 27s + 9.8}{s^2 + 5s + 6} \right|_{s=0} = 1.633$$

$$C_2 = \left. \frac{9s^2 + 27s + 9.8}{s(s+3)} \right|_{s=-2} = 4.100$$

$$C_3 = \left. \frac{9s^2 + 27s + 9.8}{s(s+2)} \right|_{s=-3} = 3.266$$

$$P(s) = \frac{1.633}{s} + \frac{4.100}{s+2} + \frac{3.266}{s+3}$$

$$p(t) = 1.633 + 4.100e^{-2t} + 3.266e^{-3t}$$

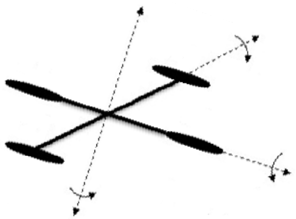


Example) $\dfrac{s+6}{s(s+1)(s+2)}$

$$\frac{s+6}{s(s+1)(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

$$\begin{cases} k_1 = \left.\dfrac{s+6}{(s+1)(s+2)}\right|_{s=0} = 3 \\ k_2 = \left.\dfrac{s+6}{s(s+2)}\right|_{s=-1} = -5 \\ k_3 = \left.\dfrac{s+6}{s(s+1)}\right|_{s=-2} = 2 \end{cases}$$

$$\therefore f(t) = \mathcal{L}^{-1}\left[\frac{3}{s} - \frac{5}{s+1} + \frac{2}{s+2}\right] = 3 - 5e^{-t} + 2e^{-2t}$$



Example) $\dfrac{2s^3+6}{(s+2)^4}$

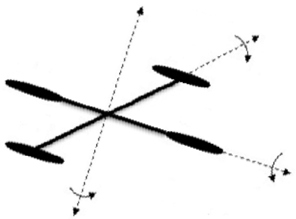
$$\frac{2s^3+6}{(s+2)^4}=\frac{k_0}{(s+2)^4}+\frac{k_1}{(s+2)^3}+\frac{k_2}{(s+2)^2}+\frac{k_3}{s+2}$$

$$\begin{cases} k_0 = 2s^3+6\Big|_{s=-2} = -10 \\ k_1 = \dfrac{d}{ds}(2s^3+6)\Big|_{s=-2} = 6s^2\Big|_{s=-2} = 24 \\ k_2 = \dfrac{1}{2!}\dfrac{d^2}{ds^2}(2s^3+6)\Big|_{s=-2} = \dfrac{1}{2}\times 12s\Big|_{s=-2} = -12 \\ k_3 = \dfrac{1}{3!}\dfrac{d^3}{ds^3}(2s^3+6)\Big|_{s=-2} = \dfrac{1}{6}\times 12\Big|_{s=-2} = 2 \end{cases}$$

$$t^n e^{-at} \Leftrightarrow \frac{(n-1)!}{(s+a)^{n+1}}$$

$$\therefore f(t) = \mathcal{L}^{-1}\left[-\frac{10}{(s+2)^4}+\frac{24}{(s+2)^3}-\frac{12}{(s+2)^2}+\frac{2}{s+2}\right]$$

$$= e^{-2t}\left[-\frac{10}{6}t^3+12t^2-24t+2\right]$$



Example) $\dfrac{4s+6}{(s+1)(s+2)^2}$

$$\frac{4s+6}{(s+1)(s+2)^2} = \frac{k_0}{(s+2)^2} + \frac{k_1}{s+2} + \frac{k_2}{s+1}$$

$$\begin{cases} k_0 = \left.\dfrac{4s+6}{s+1}\right|_{s=-2} = 2, \quad k_1 = \left.\dfrac{d}{ds}\left(\dfrac{4s+6}{s+1}\right)\right|_{s=-2} = -2 \\ k_3 = \left.\dfrac{4s+6}{(s+2)^2}\right|_{s=-1} = 2 \end{cases}$$

$$\frac{4s+6}{(s+1)(s+2)^2} = \frac{2}{(s+2)^2} - \frac{2}{s+2} + \frac{2}{s+1}$$

이므로

$$\therefore f(t) = \mathcal{L}^{-1}\left[\frac{2}{(s+2)^2} - \frac{2}{s+2} + \frac{2}{s+1}\right] = 2e^{-2t}(t-1) + 2e^{-t}$$

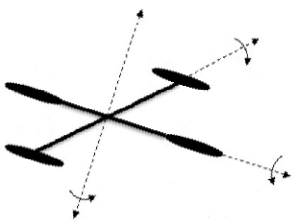
Example) $\dfrac{2s-1}{s^2+2s+5}$

$$\frac{2s-1}{s^2+2s+5}=\frac{2s-1}{(s+1)^2+4}=\frac{2(s+1)}{(s+1)^2+4}-\frac{3}{(s+1)^2+4}$$

$$\therefore f(t)=\mathcal{L}^{-1}\left[\frac{2(s+1)}{(s+1)^2+4}-\frac{3}{(s+1)^2+4}\right]=e^{-2t}(2\cos 2t-1.5\sin 2t)$$

$$e^{-at}\sin\omega t \Leftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$$

$$e^{-at}\cos\omega t \Leftrightarrow \frac{s+a}{(s+a)^2+\omega^2}$$



Solution of Derivative using Laplace Transform

정리 4.4 **도함수의 변환**

$f, f', \ldots, f^{(n-1)}$ 이 $[0, \infty)$에서 연속이고 지수적 차수를 가지며 $f^{(n)}(t)$가 $[0, \infty)$에서 조각별로 연속이라 가정하자. 그러면

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

이 성립한다. 단 $F(s) = \mathscr{L}\{f(t)\}$이다.

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathscr{L}\{f(t)\}$$

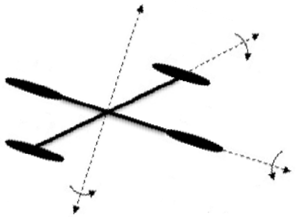
$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

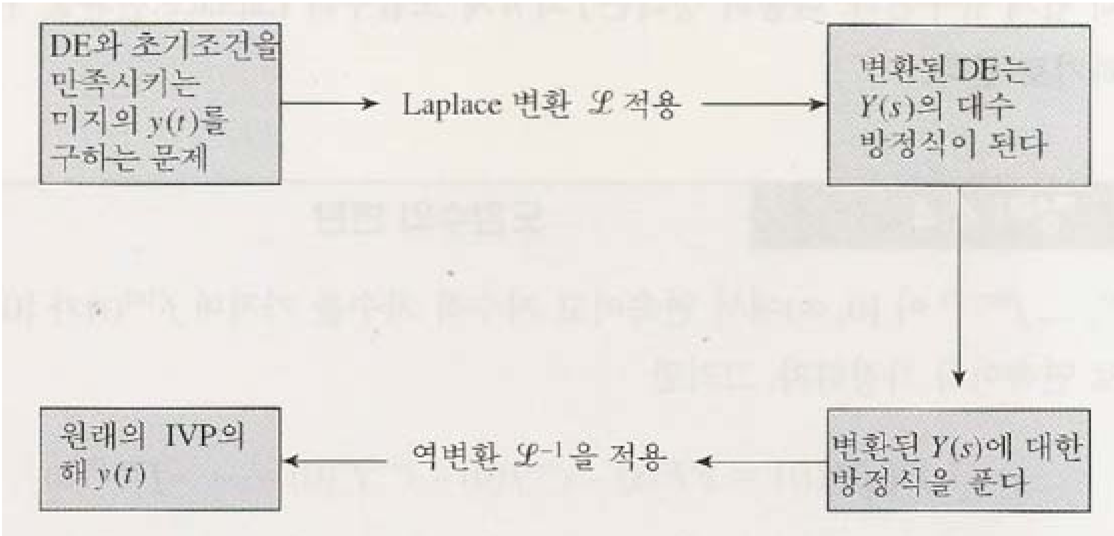
$$\mathscr{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt = e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) + s\mathscr{L}\{f'(t)\}$$

$$= s[sF(s) - f(0)] - f'(0) \quad \leftarrow (6)\text{으로 부터}$$

$$\mathscr{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$




DE와 초기조건을 만족시키는 미지의 y(t)를 구하는 문제
Laplace 변환 ℒ 적용
변환된 DE는 Y(s)의 대수 방정식이 된다
변환된 Y(s)에 대한 방정식을 푼다
역변환 ℒ⁻¹을 적용
원래의 IVP의 해 y(t)

예제 4 1계 IVP 구하기

Laplace 변환을 이용하여 주어진 초기값 문제를 풀라.

$$\frac{dy}{dt} + 3y = 13\sin 2t, \qquad y(0) = 6$$

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + 3\mathscr{L}\{y\} = 13\mathscr{L}\{\sin 2t\}$$

$\mathscr{L}\{dy/dt\} = sY(s) - y(0) = sY(s) - 6$ →

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4} \quad \text{또는} \quad (s+3)Y(s) = 6 + \frac{26}{s^2+4}$$

$$Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)} = \frac{6s^2+50}{(s+3)(s^2+4)}$$

$A = 8$

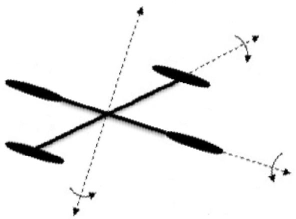
$B = -2,\ C = 6$ →

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}$$

$$Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}$$

→

$$y(t) = 8\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$



$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

예제 5 2계 IVP 구하기

다음의 초기값 문제를 풀라.

$y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2 - 3s + 2} + \frac{1}{(s^2 - 3s + 2)(s+4)} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} \qquad (14)$$

$Y(s)$를 부분분수로 분해하는 것은 예제 3에서 다루었으므로 생략하기로 한다. (4)와 (5)를 보면 초기값 문제의 해는

$$y(t) = -\frac{16}{5}e^{t} + \frac{25}{5}e^{2t} + \frac{1}{30}e^{-4t}$$

임을 알 수 있다. ❑